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A Linear Scheme for Rational Approximations, II

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Using linear difference operators, closed form rational approximations are derived for hypergeometric functions of the form ${}_pF_q(z)$, $p \leqslant q+1$, complete with error terms.

1. INTRODUCTION

We will use the following notation for gamma functions, hypergeometric functions and Meijer G-functions, respectively. If a_k, b_k, c_k are arbitrary complex parameters, s, t are complex variables and p, q, m, n are integers such that $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, we set

$$\Gamma_n(s + c_P + t) = \prod_{k=n+1}^{p} \Gamma(s + c_k + t), \qquad \Gamma(s + c_Q + t) = \Gamma_0(s + c_Q + t),$$

$$(s)_t = \frac{\Gamma(s+t)}{\Gamma(s)}, \qquad (s + c_Q)_t = \prod_{k=1}^{q} (s + c_k)_t,$$

$$\Gamma(s + c_Q^{*j} + t) = \prod_{\substack{k=1\\k \neq j}}^{q} \Gamma(s + c_k + t), \qquad (s + c_Q^{*j})_t = \prod_{\substack{k=1\\k \neq j}}^{q} (s + c_k)_t;$$

$${}_pF_q\left(\begin{matrix} a_P \\ b_Q \end{matrix} \,\middle|\, z\right) = {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_P)_k}{(b_Q)_k} \frac{z^k}{k!};$$

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_P \\ b_Q \end{matrix}\right) = G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right),$$

$$= \frac{1}{2\pi i} \int_L \frac{\Gamma(b_M - t)\,\Gamma(1 - a_N + t)\, z^t}{\Gamma_m(1 - b_Q + t)\,\Gamma_n(a_P - t)}\, dt, \tag{1.1}$$

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where L is an upward oriented contour which separates the poles of $\Gamma(b_M - t)$ from those of $\Gamma(1 - a_N + t)$, and which runs from $-i\infty$ to $+i\infty$ $(L = L_0)$ or begins and ends at $+\infty$ $(L = L_+)$ or $-\infty$ $(L = L_-)$.

The basic functional relationships for the G-function are then

$$G^{m,n}_{p,q}\left(w \,\middle|\, \begin{matrix} a_P \\ b_Q \end{matrix}\right) = G^{n,m}_{q,p}\left(w^{-1} \,\middle|\, \begin{matrix} 1 - b_Q \\ 1 - a_P \end{matrix}\right),$$
$$w^c G^{m,n}_{p,q}\left(w \,\middle|\, \begin{matrix} a_P \\ b_Q \end{matrix}\right) = G^{m,n}_{p,q}\left(w \,\middle|\, \begin{matrix} c + a_P \\ c + b_Q \end{matrix}\right). \tag{1.2}$$

If the poles of the integrand in (1.1), interior to L, are simple, then the G-function is a finite sum of hypergeometric functions, e.g.,

$$G^{m,n}_{p,q}\left(w \,\middle|\, \begin{matrix} a_P \\ b_Q \end{matrix}\right) = \sum_{k=1}^{m} \frac{\Gamma(b_M^{*k} - b_k)\,\Gamma(1 - a_N + b_k)}{\Gamma(1 - b_Q + b_k)\,\Gamma(a_P - b_k)}\, w^{b_k}$$
$$\times {}_{p+1}F_q\left(\begin{matrix} 1, 1 - a_P + b_k \\ 1 - b_Q + b_k \end{matrix} \,\middle|\, (-1)^{p-m-n} w\right), \tag{1.3}$$
$$p < q, \text{ or } p = q \text{ and } |w| < 1.$$

For a more detailed discussion of hypergeometric functions and Meijer G-functions, see [1, 2].

In [5], a linear scheme for rational approximation was introduced and, as an application, closed form rational approximations were developed for the Meijer G-functions,

$$F(v) = \frac{\Gamma(\beta_Q)}{\Gamma(\alpha_P)}\, G^{1,p}_{p,q+1}\left(v^{-1} \,\middle|\, \begin{matrix} 1 - \alpha_P \\ 0, 1 - \beta_Q \end{matrix}\right), \qquad p > q + 1;$$
$$= \frac{1}{2\pi i} \int_{L-} \frac{\Gamma(-s)(\alpha_P)_s}{(\beta_Q)_s}\, v^{-s}\, ds,$$
$$\sim {}_pF_q\left(\begin{matrix} \alpha_P \\ \beta_Q \end{matrix} \,\middle|\, \frac{-1}{v}\right), \qquad v \to \infty, |\arg v| < \pi(p + 1 - q)/2,$$

under mild restrictions on the parameters α_r, β_k. In particular, with

$$H_n(v) = {}_{q+2}F_p\left(\begin{matrix} -n, n + \lambda, -a + \beta_Q \\ -a + 1 + \alpha_P \end{matrix} \,\middle|\, -v\right), \qquad K_n(v) = K_n(v, v), \tag{1.4}$$

$$K_n(v, \gamma) = \sum_{k=a}^{n} (-\gamma)^k \sum_{j=0}^{n-k} \frac{(-n)_{k+j}(n + \lambda)_{k+j}(-a + \beta_Q)_{k+j}(\alpha_P)_j}{(-a + 1 + \alpha_P)_{k+j}(\beta_Q)_j(k + j)!\, j!}\, (\gamma/v)^j,$$
$$a\ (\geqslant 0),\ \lambda \text{ integers},\ a + \lambda \leqslant p, \tag{1.5}$$

it was shown in [5] that

(1) $F(v) = \lim_{n\to\infty} K_n(v)/H_n(v)$, v $(\neq 0)$ fixed, $|\arg v| < \pi$,

(2) as for the Padé approximants to $F(v)$, $K_n(v)$ and $H_n(v)$ satisfy the same homogeneous difference equation with respect to n, and

(3) the error, $F(v) - K_n(v)/H_n(v)$, can be represented by an easily analyzed, closed form expression.

These results were derived by series manipulation of entire functions.

In this paper, we derive similar results when

$$F(v) = \frac{\Gamma(\beta_Q)}{\Gamma(\alpha_P)} G^{1,p}_{p,q+1}\left(v^{-1} \,\middle|\, \begin{matrix} 1-\alpha_P \\ 0, 1-\beta_Q \end{matrix}\right), \qquad p \leqslant q+1; \tag{1.6}$$

$$= \frac{1}{2\pi i}\int_{L_+} \frac{\Gamma(-s)(\alpha_P)_s}{(\beta_Q)_s} v^{-s}\, ds, \qquad (|v| > 1 \text{ if } p = q+1);$$

$$= {}_pF_q\left(\begin{matrix}\alpha_P \\ \beta_Q\end{matrix} \,\middle|\, \frac{-1}{v}\right). \tag{1.7}$$

The results for $p < q+1$ will follow from those for $p = q+1$, while the results for $p = q+1$ will be derived by manipulation of series with finite radii of convergence, and analytic continuation. Partial results for $p \leqslant q+1$ are contained in [2–4]. As the case $p = q+1$ is of central importance, we will subsequently refer to it as the central case, and will discuss, $F(v)$ for this central case in more detail.

It follows from the general theory of hypergeometric functions and Meijer G-functions that in the central case $F(v)$ satisfies a linear differential equation $\mathscr{H}y = 0$ of order $p = q+1$, whose only singularities are the regular singular points at $v = 0$, -1 and ∞. Thus, in a neighborhood of each of these singular points, $F(v)$ has a local representation as a linear combination of an appropriate basis of $\mathscr{H}y = 0$. We take $F(v)$ to be initially defined in $|v| > 1$ by the ${}_{q+1}F_q(-1/v)$ in (1.7). The Meijer G-function in (1.6) then serves to analytically continue $F(v)$ into $|v| \leqslant 1$. A local representation about $v = 0$ can be deduced from (1.2) and (1.3). Finally, it follows from the general theory of hypergeometric functions, Nørlund [7], that $F(v)$ has a local representation at $v = e^{-i\pi}$ of the form

$$F(v) = \frac{\Gamma(-\sigma)\,\Gamma(\beta_Q)}{\Gamma(\alpha_{Q+1})}\left[1 + \frac{1}{v}\right]^{\sigma} R_1(v) + R_2(v), \qquad |1 + v^{-1}| < 1,$$

$$\arg(e^{i\pi}v_0) = \arg(1 + v_0^{-1}) = 0, \qquad v_0 = \left(\frac{4}{3}\right)e^{-i\pi}, \qquad R_1(-1) = 1, \tag{1.8}$$

$$\sigma = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{q+1} \alpha_j \neq \text{an integer},$$

where $R_1(v)$ and $R_2(v)$ are analytic functions of v in $|1+v^{-1}|<1$. Similar expansions involving $\log(1+v^{-1})$ exist when σ is an integer.

PROPOSITION. *In the central case, $p=q+1$, let $F(v)$ be defined initially for $|v|>1$ by* (1.7), *and analytically continued into $|v|\leqslant 1$ by* (1.6). *Also, let $F(ve^{i2\pi})$ be $F(v)$ analytically continued along a path which encloses $v=0$, but not $v=-1$. Then*

$$F(e^{i\pi})=F(e^{-i\pi}), \qquad \operatorname{Re}\sigma>0, \tag{1.9}$$

where σ is defined in (1.8).

Proof. Consider the homotopic paths Γ and Γ' in Fig. 1 connecting v and $ve^{i2\pi}$. Let $v=(1+\varepsilon)\,e^{-i\pi}$, $0<\varepsilon<\frac{1}{4}$, so that the points C and D coincide. As the initial branch of $F(v)$ is single valued in $|v|>1$, $F(v)$ takes on the same value at the points A and B. Representing $F(v)$ for v near $e^{-i\pi}$ by (1.8), we have for $\sigma\neq$ an integer,

$$F((1+\varepsilon)\,e^{-i\pi})=\frac{\Gamma(-\sigma)\,\Gamma(\beta_Q)}{\Gamma(\alpha_{Q+1})}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{\sigma}R_1(-1+\varepsilon e^{i\pi})+R_2(-1+\varepsilon e^{i\pi}),$$

$$F((1+\varepsilon)\,e^{i\pi})=\frac{\Gamma(-\sigma)\,\Gamma(\beta_Q)}{\Gamma(\alpha_{Q+1})}\left(\frac{\varepsilon e^{-i2\pi}}{1+\varepsilon}\right)^{\sigma}R_1(-1+\varepsilon e^{i\pi})+R_2(-1+\varepsilon e^{i\pi}).$$

Letting $\varepsilon\to 0$ in these equations, we arrive at $F(e^{i\pi})=F(e^{-i\pi})$. A similar proof holds when σ is a positive integer. ∎

The following asymptotic estimates were established in [6].

THEOREM 1. *Let $q+1$, $a+1$ be positive integers, n, λ, α_r $(r=1,\ldots,q+2;\ \alpha_{q+2}=a)$, β_k $(k=1,\ldots,q)$ be complex numbers such that n is large,*

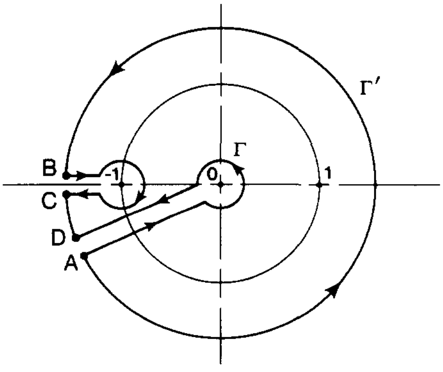

FIG. 1. v-plane, v at A, $ve^{i2\pi}$ at D.

$|n| \to +\infty$, $\arg n = \mathcal{O}(n^{-1})$ *as* $n \to \infty$, *and* $\lambda, \alpha_r, \beta_k$ *be bounded with respect to* n. *Then*

$$G_n(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G^{q+2,1}_{q+2,q+2}\left(w \,\middle|\, \begin{matrix} 1-n-\lambda, a+1-\beta_Q, n+1 \\ a-\alpha_{Q+2} \end{matrix}\right),$$

$$= \sqrt{\pi}[n^2 w]^{\tau}(1+w)^{-\tau-\lambda/2} e^{-(2n+\lambda)\xi}[1+\mathcal{O}((nw)^{-1})], \qquad nw \to \infty, \tag{1.10}$$

$$|\arg w| \leqslant 3\pi - \varepsilon, \qquad |\arg(1+w)| \leqslant 2\pi - \varepsilon, \qquad |w(1+w)| \geqslant \varepsilon,$$

$$\cosh 2\xi = 1 + 2w, \qquad 2\tau = a + \frac{1}{2} + \sum_{k=1}^{q} \beta_k - \sum_{r=1}^{q+1} (1+\alpha_r),$$

where ε *is a small positive number independent of* n, *and* $\arg \xi = 0$ *for* $\arg w = \arg(1+w) = 0$. *Also, if* $G_n(ve^{i2\pi})$ *is* $G_n(v)$ *analytically continued along a curve which encloses* $v = 0$, *but not* $v = -1$, *then with* $v = e^{-i\pi}$,

$$e^{-i2\pi\tau} G_n(e^{i\pi}) + e^{i2\pi\tau} G_n(e^{-i\pi})$$

$$= \frac{2\pi(-1)^n}{\Gamma(1/2 + 2\tau + \lambda)} \left(n + \frac{\lambda}{2}\right)^{4\tau+\lambda} \{1 + \mathcal{O}(n^{-2})\}, \tag{1.11}$$

$$n \to \infty, \ n = \text{positive integer}.$$

Moreover, if

$$\begin{aligned} &-1 + \beta_k - \alpha_r \neq \text{a negative integer}, && k = 1,\dots, q, \quad r = 1,\dots, q+2; \\ &\beta_k - \beta_j \neq \text{an integer}, && k \neq j, \quad k, j = 1,\dots, q; \end{aligned} \tag{1.12}$$

then

$$L_{n,k}(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)}$$

$$\times G^{q+3,2}_{q+3,q+3}\left(w \,\middle|\, \begin{matrix} 1-n-\lambda, a+1-\beta_k, a+1-\beta_Q, n+1 \\ a-\alpha_{Q+2}, a+1-\beta_k \end{matrix}\right),$$

$$= \frac{\Gamma(-\alpha_{Q+2} + \beta_k)}{\Gamma(1-\beta_Q+\beta_k)} [n^2 w]^{a-\beta_k}\{1 + \mathcal{O}((n^2 w)^{-1})\}, \tag{1.13}$$

$$n^2 w \to \infty, \qquad |\arg w| \leqslant 2\pi - \varepsilon, \qquad k = 1,\dots, q.$$

Remark 1. The asymptotic expansion of a suitably weighted linear combination of the $L_{n,k}(w)$, say, $L(w)$, can be deduced from (1.13) even when the parameter restrictions (1.12) are violated, provided suitable limits are taken. At worst, these limits introduce positive integer powers of $\log |n^2 w|$ into (1.13), and hence into the asymptotic expansion of $L(w)$.

2. Central Case, $p = q + 1$

THEOREM 2. *Let* $q+1$, $n+1$, $a+1$, $\lambda+1$ *be positive integers*, α_r ($r = 1,\ldots, q+2$; $\alpha_{q+2} = a$), β_k ($k = 1,\ldots, q$) *be complex numbers such that*

$$\begin{aligned} &a + \lambda = \text{an integer} \leqslant q + 1; \\ &-1 + \beta_k, -1 - a + \beta_k \neq \text{a negative integer}, \qquad k = 1,\ldots, q; \\ &-1 + \alpha_k, -a + \alpha_k \neq \text{a negative integer}, \qquad k = 1,\ldots, q+1; \end{aligned} \tag{2.1}$$

and set

$$H_n(v) = {}_{q+2}F_{q+1}\left(\begin{matrix} -n, n+\lambda, -a+\beta_Q \\ -a+1+\alpha_{Q+1} \end{matrix} \,\middle|\, -v\right),$$

$$K_n(v) = \sum_{k=a}^{n} (-v)^k \sum_{j=0}^{n-k} \frac{(-n)_{k+j}(n+\lambda)_{k+j}(-a+\beta_Q)_{k+j}(\alpha_{Q+1})_j}{(-a+1+\alpha_{Q+2})_{k+j}(\beta_Q)_j j!}.$$

Then for v fixed,

$$\begin{aligned} \lim_{n\to\infty} \frac{K_n(v)}{H_n(v)} &= {}_{q+1}F_q\left(\begin{matrix} \alpha_{Q+1} \\ \beta_Q \end{matrix} \,\middle|\, \frac{-1}{v}\right), \qquad |v| > 1; \\ &= \frac{\Gamma(\beta_Q)}{\Gamma(\alpha_{Q+1})} G^{1,q+1}_{q+1,q+1}\left(v^{-1} \,\middle|\, \begin{matrix} 1-\alpha_{Q+1} \\ 0, 1-\beta_Q \end{matrix}\right), \qquad v \in \mathscr{D}, \\ &\mathscr{D} = \{v\colon |\arg v| \leqslant \pi, |\arg(1+v)| \leqslant \pi, v \notin [-1, 0]\}. \end{aligned} \tag{2.2}$$

Moreover, the convergence is uniform on closed subsets of $\mathscr{D}$. *Finally, if*

$$\lambda - 1 - 2q + 2\,\mathrm{Re}\left\{\sum_{j=1}^{q} \beta_j - \sum_{j=1}^{q+1} \alpha_j\right\} > -2\,\mathrm{Re}\,\beta_k, \qquad k = 1,\ldots, q, \tag{2.3}$$

then (2.2) *is also valid for* $v = e^{\pm i\pi}$.

Proof. As the initial branch of $F(v)$ and the polynomials $H_n(v)$, $K_n(v)$ are single-valued in $|v| > 1$, it is sufficient to establish the theorem in the restricted region.

$$\mathscr{D}^+ = \{v\colon -\pi \leqslant \arg v < \pi, -\pi \leqslant \arg(1+v) < \pi, v \notin [-1, 0]\}, \tag{2.4}$$

and indicate the minor changes in the analysis for the region $\mathscr{D}^- = \overline{\mathscr{D}^+}$, the complex conjugate of $\mathscr{D}^+$. Also, in addition to the parameter restrictions (2.1) of the theorem, we will tentatively assume

$$\begin{aligned} &\beta_k - \beta_j \neq \text{an integer}, \qquad k \neq j, \quad k, j = 1,\ldots, q; \\ &\alpha_r - \alpha_j \neq \text{an integer}, \qquad r \neq j, \quad r, j = 1,\ldots, q+1. \end{aligned} \tag{2.5}$$

Under these conditions, Theorems 3 and 4 analyze the asymptotic behaviour of $H_n(v)$ and $K_n(v)$, respectively, in $\mathscr{D}^+$ as $n \to \infty$. Theorem 2 then follows directly. ∎

THEOREM 3. *Under the conditions of Theorem 2 and (2.5), there exist numbers A_1, A_2 and B_k, $k = 1,\ldots, q$, such that*

$$\frac{\Gamma(-a+1+\alpha_{Q+1})}{\Gamma(-a+\beta_Q)} \frac{\Gamma(s-a+\beta_Q)\,\Gamma(1-s+a-\beta_Q)}{\Gamma(s-a+1+\alpha_{Q+1})\,\Gamma(-s+a-\alpha_{Q+1})} = \sum_{k=1}^{q} B_k \Gamma(s-a+\beta_k)\,\Gamma(1-s+a-\beta_k) + A_1 e^{i\pi s} + A_2 e^{-i\pi s}, \tag{2.6}$$

and

$$H_n(v) = {}_{q+2}F_{q+1}\left(\begin{matrix}-n, n+\lambda, -a+\beta_Q \\ -a+1+\alpha_{Q+1}\end{matrix}\,\middle|\, ve^{i\pi}\right) = \sum_{k=1}^{q} B_k L_{n,k}(ve^{i\pi}) + A_1 G_n(ve^{i2\pi}) + A_2 G_n(v), \tag{2.7}$$

where $G_n(ve^{i2\pi})$ is $G_n(v)$ analytically continued along a curve which encloses $v = 0$, but not $v = -1$ so that $\arg(1 + ve^{i2\pi}) = \arg(1 + v)$. *In particular,*

$$B_k = \frac{\Gamma(-a+1+\alpha_{Q+1})\,\Gamma(\beta_k+1-\beta_Q)\,\Gamma(-\beta_k+\beta_Q^{*k})}{\Gamma(-a+\beta_Q)\,\Gamma(\beta_k-\alpha_{Q+1})\,\Gamma(-\beta_k+1+\alpha_{Q+1})}, \qquad k = 1,\ldots, q,$$

$$e^{i2\pi\tau}A_1 = e^{-i2\pi\tau}A_2 = A_0 = \frac{\Gamma(-a+1+\alpha_{Q+1})}{2\pi\Gamma(-a+\beta_Q)},$$

$$2\tau = a - \frac{1}{2} - q + \sum_{k=1}^{q} \beta_k - \sum_{r=1}^{q+1} \alpha_r = a - \frac{1}{2} - q + \sigma.$$

For $v \in \mathscr{D}^+$ as defined in (2.4),

$$\begin{aligned} H_n(v) &\sim A_1 G_n(ve^{i2\pi}), && nv \to \infty, \\ &\sim \frac{\Gamma(-a+1+\alpha_{Q+1})}{2\sqrt{\pi}\,\Gamma(-a+\beta_Q)} [n^2 v]^{\tau} (1+v)^{-\tau-\lambda/2} e^{(2n+\lambda)\xi}, && nv \to \infty, \end{aligned} \tag{2.8}$$

$$\cosh 2\xi = 1 + 2v, \qquad \operatorname{Re} \xi > 0,$$

where n, λ satisfy the restrictions of Theorem 1. *With the same notation,*

$$H_n(-1) \sim A_0[e^{-i2\pi\tau}G_n(e^{i\pi}) + e^{i2\pi\tau}G_n(e^{-i\pi})], \qquad n \to \infty,$$

$$\sim \frac{(-1)^n\,\Gamma(-a+1+\alpha_{Q+1})}{\Gamma(1/2+2\tau+\lambda)\,\Gamma(-a+\beta_Q)}(n+\lambda/2)^{4\tau+\lambda}, \qquad n \to \infty, \quad (2.9)$$

n a positive integer; $\quad \mathrm{Re}(4\tau+\lambda) > \mathrm{Re}(2a-2\beta_k), \quad k = 1,..., q.$

Proof. The identity (2.6) is just a paraphrase of the partial fraction decomposition, $y = e^{i2\pi s}$,

$$\frac{\Gamma(-a+1+\alpha_{Q+1})}{\Gamma(-a+\beta_Q)}\frac{\prod_{r=1}^{q+1}(y-e^{i2\pi(a-\alpha_r)})}{\prod_{k=1}^{q}(y-e^{i2\pi(a-\beta_k)})}$$

$$= \sum_{k=1}^{q}\frac{d_k y}{y-e^{i2\pi(a-\beta_k)}} + c_1 y + c_0.$$

For $|v| < 1$, $H_n(v)$ has the Mellin–Barnes integral representation

$$H_n(v) = \frac{1}{2\pi i}\int_{L_+}\frac{\Gamma(-s)(-a+\beta_Q)_s(n+\lambda)_s(ve^{i\pi})^s}{(-a+1+\alpha_{Q+1})_s(n+1)_{-s}}\,ds.$$

Substituting identity (2.6) into this representation and identifying the resulting integrals, we arrive at (2.7) for $|v| < 1$. By analytic continuation, (2.7) then holds for all v. In particular, setting $v = e^{-i\pi}$ in (2.7) gives the representation for $H_n(-1)$,

$$H_n(-1) = \sum_{k=1}^{q} B_k L_{n,k}(1) + A_0[e^{-i2\pi\tau}G_n(e^{i\pi}) + e^{i2\pi\tau}G_n(e^{-i\pi})]. \quad (2.10)$$

Replacing the various functions in (2.7) and (2.10) by their asymptotic expansions as given in Theorem 1, and picking out the dominant terms of the resulting expressions, we arrive at (2.8) and (2.9). Note that the algebraic condition in (2.9) for $4\tau + \lambda$ is the same as the algebraic condition (2.3). Also, note that ξ as defined in (1.10) is a function of w, i.e., $\xi = \xi(w)$, and that for $|w| < 1$,

$$\xi(we^{i\pi}) = -\xi(we^{-i\pi}).$$

By analytic continuation, this relation continues to hold for all w, and with $w = ve^{i\pi}$,

$$\mathrm{Re}\{\xi(ve^{i2\pi})\} < 0 < \mathrm{Re}\{\xi(v)\}, \qquad v \in \mathscr{D}^+. \quad \blacksquare$$

Remark 2. Equations (2.7) and (2.8) remain true even when n is not a positive integer, although in that case $H_n(v)$ is not a polynomial, and (2.7) then serves to analytically continue $H_n(v)$ into $|v| \geqslant 1$.

THEOREM 4. *Under the conditions of Theorem* 2 *and* (2.5), *and with the notation of Theorems* 2 *and* 3,

$$K_n(v) = \sum_{k=1}^{q} \frac{B_k E_k(v)}{\Gamma(\beta_k)\,\Gamma(1-\beta_k)} L_{n,k}(ve^{i\pi}) + A_1 F(v)\, G_n(ve^{i2\pi}) + A_2 F(ve^{i2\pi})\, G_n(v), \tag{2.11}$$

$$E_k(v) = \frac{\Gamma(\beta_Q)}{\Gamma(\alpha_{Q+1})} G_{q+2,q+2}^{q+2,2}\left(ve^{i\pi} \,\middle|\, \begin{matrix} 1, \beta_k, \beta_Q \\ \beta_k, \alpha_{Q+1} \end{matrix}\right),$$

where $G_n(ve^{i2\pi})$, $F(ve^{i2\pi})$ *are* $G_n(v)$, $F(v)$, *respectively, analytically continued along a curve which encloses* $v = 0$, *but not* $v = -1$. *For* $v \in \mathscr{S}^+$

$$K_n(v) \sim A_1 F(v)\, G_n(ve^{i2\pi}), \qquad nv \to \infty. \tag{2.12}$$

With the same notation as in Theorem 1,

$$K_n(-1) \sim F(-1) A_0 [e^{i2\pi\tau} G_n(e^{i\pi}) + e^{i2\pi\tau} G_n(e^{-i\pi})], \qquad n \to \infty,$$
$$\mathrm{Re}(4\tau + \lambda) > \mathrm{Re}(2a - 2\beta_k), \qquad k = 1,..., q.$$

Proof. It is sufficient to establish (2.11) for $|v| < 1$, as all the functions involved can be analytically continued into the region $|v| \geqslant 1$. In particular, (2.11) is valid for $v = e^{-i\pi}$. The asymptotic estimates then follow directly from Theorem 1.

Let $I_k(\gamma/v)$ be defined by (1.5) and

$$K_n(v, \gamma) = \sum_{k=a}^{n} \gamma^k I_k(\gamma/v).$$

Then for $|\gamma| \leqslant |v|$,

$$\frac{\Gamma(n+\lambda)(\beta_Q)_{-a}}{\Gamma(n+1)(\alpha_{Q+1})_{-a+1}} I_k(\gamma/v) = G_{2q+3,2q+3}^{1,2q+2}\left(\frac{\gamma}{v} \,\middle|\, \begin{matrix} 1-\alpha_{Q+1}, a+1-\beta_Q-k, 1-n-\lambda-k, n+1-k \\ 0, 1-\beta_Q, a-\alpha_{Q+2}-k \end{matrix}\right),$$

where L, the contour of integration in the integral definition of the $G_{2q+3,2q+3}^{1,2q+2}(\gamma/v)$ function, is of the form L_+. When $\gamma = v$, this contour L can be deformed successively into contours of the form L_0 and L_-. Evaluating the final integral by the residue calculus, we obtain a result formally indicated by setting $w = 1$ in (1.2) and (1.3), i.e.,

$$\frac{\Gamma(n+\lambda)(\beta_Q)_{-a}}{\Gamma(n+1)(\alpha_{Q+1})_{-a+1}} K_n(v, v) \tag{2.13}$$
$$= \sum_{k=a}^{n} v^k G^{2q+2,1}_{2q+3,2q+3}\left(1 \,\middle|\, \begin{matrix} 1, \beta_Q, -a+1+\alpha_{Q+2}+k \\ \alpha_{Q+1}, -a+\beta_Q+k, n+\lambda+k, -n+k \end{matrix}\right).$$

For $|v| < 1$, let $F_r(v)$ and $G_{n,r}(v)$ be defined by

$$F(v) = \sum_{r=1}^{q+1} v^{\alpha_r} F_r(v), \qquad G_n(v) = \sum_{r=1}^{q+2} v^{a-\alpha_r} G_{n,r}(v). \tag{2.14}$$

Explicit expressions for the $F_r(v)$ and $G_{n,r}(v)$ can be deduced using (1.3). In particular, $\alpha_{q+2} = a$, and

$$\Gamma(a+1-\beta_Q)\, G_{n,q+2}(v) = \Gamma(a-\alpha_{Q+1})\, H_n(v),$$

where $H_n(v)$ is defined in Theorem 3. Comparing the coefficients of v^k in (2.13) with those in $v^a F_r(v)\, G_{n,r}(v)$, we see that when $a+\lambda$ is an integer $\leqslant q+1$,

$$K_n(v) = K_n(v, v), \tag{2.15}$$
$$= \frac{\Gamma(-a+1+\alpha_{Q+1})}{\Gamma(-a+\beta_Q)} \sum_{r=1}^{q+1} \frac{\Gamma(\beta_Q-\alpha_r)\,\Gamma(1-\beta_Q+\alpha_r)\, v^a F_r(v)\, G_{n,r}(v)}{\Gamma(-\alpha^{*r}_{Q+2}+\alpha_r)\,\Gamma(1+\alpha^{*r}_{Q+2}-\alpha_r)}.$$

For more details, see [5, Theorem 4].

Next, consider the identity

$$\frac{\Gamma(-a+1+\alpha_{Q+1})}{\Gamma(-a+\beta_Q)} e^{i\pi(s-t)}\Gamma(t+a-s)\,\Gamma(1-t-a+s)$$
$$\times \left\{\frac{\Gamma(t+\beta_Q)\,\Gamma(1-t-\beta_Q)}{\Gamma(t+1+\alpha_{Q+1})\,\Gamma(-t-\alpha_{Q+1})\,\Gamma(t)\,\Gamma(1-t)}\right.$$
$$\left. - \frac{\Gamma(s-a+\beta_Q)\,\Gamma(1-s+a-\beta_Q)}{\Gamma(s-a)\,\Gamma(1-s+a)\,\Gamma(s-a+1+\alpha_{Q+1})\,\Gamma(-s+a+\alpha_{Q+1})}\right\}$$
$$= \sum_{k=1}^{q} B_k e^{i\pi(s-t)} \frac{\Gamma(t+\beta_k)\,\Gamma(1-t-\beta_k)\,\Gamma(s-a+\beta_k)\,\Gamma(1-s+a-\beta_k)}{\Gamma(\beta_k)\,\Gamma(1-\beta_k)}$$
$$+ A_1 e^{i\pi 2s} + A_2 e^{-i\pi 2t}, \tag{2.16}$$

where A_1, A_2, B_k are defined in Theorem 3. To prove this identity, replace s by $t+a$ in (2.6), and subtract this expression from (2.6). After some manipulation, one arrives at (2.16). Note that $s=t+a$ is only an apparent singularity of (2.16). Let $|\gamma|<1$, and $\mathscr{L}_+$ be an s contour of the form L_+, which separates the poles of $\Gamma(-s+a-\alpha_{Q+2})$ from those of $\Gamma(s+n+\lambda)\,\Gamma(s-a+\beta_Q)$. Then multiplying (2.16) by

$$\frac{1}{2\pi i}\,\frac{\Gamma(-s+a-\alpha_{Q+2})(n+\lambda)_s}{\Gamma(-s+a+1-\beta_Q)(n+1)_{-s}}\,\gamma^s$$

and integrating the resulting identity with respect to s over $\mathscr{L}_+$, we obtain the identity

$$\begin{aligned}
&\frac{\Gamma(-a+1+\alpha_{Q+1})\,\Gamma(t+\beta_Q)\,\Gamma(1-t-\beta_Q)\,e^{-i\pi t}}{\Gamma(-a+\beta_Q)\,\Gamma(t+1+\alpha_{Q+1})\,\Gamma(-t-\alpha_{Q+1})\,\Gamma(t)\,\Gamma(1-t)}\\
&\qquad\times\sum_{r=1}^{q+1}(\gamma e^{i\pi})^{a-\alpha_r}\,\Gamma(t+\alpha_r)\,\Gamma(1-t+\alpha_r)\,G_{n,r}(\gamma)\\
&\qquad+\frac{\Gamma(-a+1+\alpha_{Q+1})\,\Gamma(a-\alpha_{Q+1})\,\Gamma(t+a)\,\Gamma(1-t-a)}{\Gamma(-a+\beta_Q)\,\Gamma(1+a-\beta_Q)}\,e^{-i\pi t}H_n(\gamma)\\
&\qquad\times\left\{\frac{\Gamma(t+\beta_Q)\,\Gamma(1-t-\beta_Q)}{\Gamma(t+1+\alpha_{Q+1})\,\Gamma(-t-\alpha_{Q+1})\,\Gamma(t)\,\Gamma(1-t)}\right.\\
&\qquad\left.-\frac{\Gamma(-a+\beta_Q)\,\Gamma(1+a-\beta_Q)}{\Gamma(1-a+\alpha_{Q+1})\,\Gamma(a-\alpha_{Q+1})\,\Gamma(-a)\,\Gamma(1+a)}\right\}\\
&\quad=\sum_{k=1}^{q}B_k e^{-i\pi t}\,\frac{\Gamma(t+\beta_k)\,\Gamma(1-t-\beta_k)}{\Gamma(\beta_k)\,\Gamma(1-\beta_k)}\,L_{n,k}(\gamma e^{i\pi})\\
&\qquad+A_1G_n(\gamma e^{i2\pi})+A_2e^{-i2\pi t}G_n(\gamma).
\end{aligned}\tag{2.17}$$

Note that $t=-a$ is only an apparent singularity of (2.17). Let $|v|<1$, and $\mathscr{L}_-$ be a t contour of the form L_-, which separates the poles of $\Gamma(t+\alpha_{Q+1})$ from those of $\Gamma(-t)\,\Gamma(-t+1-\beta_Q)$. Then multiplying (2.17) by

$$\frac{1}{2\pi i}\,\frac{\Gamma(\beta_Q)\,\Gamma(-t)\,\Gamma(t+\alpha_{Q+1})}{\Gamma(\alpha_{Q+1})\,\Gamma(t+\beta_Q)}\,v^{-t}$$

and integrating the resulting identity with respect to t over $\mathscr{L}_-$, we obtain the identity, $a=$ an integer $\geqslant 0$,

$$\frac{\Gamma(-a+1+\alpha_{Q+1})}{\Gamma(-a+\beta_Q)} \sum_{r=1}^{q+1} \frac{\Gamma(\beta_Q-\alpha_r)\,\Gamma(1-\beta_Q+\alpha_r)\,\gamma^a G_{n,r}(\gamma)\,F_r(v)}{\Gamma(-\alpha_{Q+2}^{*r}+\alpha_r)\,\Gamma(1+\alpha_{Q+2}^{*r}-\alpha_r)} \left(\frac{v}{\gamma}\right)^{\alpha_r}$$
$$= \sum_{k=1}^{q} \frac{B_k E_k(v)}{\Gamma(\beta_k)\,\Gamma(1-\beta_k)} L_{n,k}(\gamma e^{i\pi}) + A_1 F(v)\, G_n(\gamma e^{i2\pi}) + A_2 F(v e^{i2\pi})\, G_n(\gamma). \tag{2.18}$$

Equation (2.11) then follows from (2.15) and (2.18) with $\gamma = v$. ∎

Remark 3. If (i) is replaced by $(-i)$ in (2.7), (2.8), (2.11) and (2.12), one obtains results valid in $\mathscr{D}^-$. The tentative assumptions (2.5) can be relaxed completely by taking limits in (2.7), (2.10) and (2.11), and noticing that since this limit process introduces, at worst, positive integer powers of $\log|n^2 v|$ into the $L_{n,k}(ve^{i\pi})$ terms, the terms in (2.8), (2.9) and (2.12) are still dominant.

Remark 4. The parameter restrictions in Theorem 2 are not excessive, as

$$ {}_pF_q\left(\begin{matrix}\alpha_P\\ \beta_Q\end{matrix}\,\Big|\, z\right) = \sum_{k=0}^{m-1} \frac{(\alpha_P)_k}{(\beta_Q)_k}\cdot\frac{z^k}{k!} + \frac{(\alpha_P)_m\, z^m}{(\beta_Q)_m\, m!}\, {}_{p+1}F_{q+1}\left(\begin{matrix}1, m+\alpha_P\\ m+1, m+\beta_Q\end{matrix}\,\Big|\, z\right),$$
$$p \leqslant q+1 \qquad (|z|\leqslant 1 \text{ if } p=q+1).$$

Thus, for m sufficiently large, the restrictions (2.3) are satisfied.

The rate of convergence in (2.2) can be characterized as follows.

THEOREM 5. *Under the conditions of Theorem* 2, *and* (2.5), *and with the notation of Theorems* 1–4, *we have*

$$S_n(v) = F(v)\,H_n(v) - K_n(v) = \sum_{k=1}^{q} B_k \left[F(v) - \frac{E_k(v)}{\Gamma(\beta_k)\,\Gamma(1-\beta_k)}\right] L_{n,k}(ve^{i\pi})$$
$$+ \left[A_1 F(v) + A_2 F(v) + (-1)^a \sum_{k=1}^{q} B_k E_k(v)\right] G_n(v), \tag{2.19}$$

which reduces for $v = e^{-i\pi}$ *to*

$$S_n(-1) = \sum_{k=1}^{q} B_k \left[F(-1) - \frac{E_k(e^{-i\pi})}{\Gamma(\beta_k)\,\Gamma(1-\beta_k)}\right] L_{n,k}(1), \qquad \operatorname{Re}\sigma > 0. \tag{2.20}$$

For n large,

$$S_n(v) \sim H_n(v)\, 2\sqrt{\pi}(1+v)^{\tau+\lambda/2}\, e^{-(2n+\lambda)\xi}$$
$$\times \sum_{k=1}^{q} [\Gamma(\beta_k)\,\Gamma(1-\beta_k)\,F(v) - E_k(v)]\, \frac{e^{-i\pi\beta_k}\Gamma(-\beta_k+\beta_Q^{*k})}{\Gamma(-\beta_k+1+\alpha_{Q+2})}\, [n^2 v]^{a-\tau-\beta_k},$$
$$nv \to \infty, v \in \mathscr{D}^+; \tag{2.21}$$

$$S_n(-1) \sim H_n(-1)(-1)^{n+a}\,\Gamma(\tfrac{1}{2}+2\tau+\lambda)$$
$$\times \sum_{k=1}^{q} [\Gamma(\beta_k)\,\Gamma(1-\beta_k) - E_k(e^{-i\pi})]\, \frac{\Gamma(-\beta_k+\beta_Q^{*k})\, n^{2a-4\tau-\lambda-2\beta_k}}{\Gamma(-\beta_k+1+\alpha_{Q+2})},$$
$$n \to \infty, \qquad \mathrm{Re}(4\tau+\lambda) > \mathrm{Re}(2a-2\beta_k), \qquad k = 1,\ldots, q. \tag{2.22}$$

For $|v| > 1$, $|\arg v| \leqslant \pi$,

$$S_n(v) = \sum_{k=1}^{q} v^{a-1} F_k^*(v)\, M_{n,k}(v) + \frac{\Gamma(a+1-\beta_Q)}{\Gamma(a-\alpha_{Q+1})} F(v)\, G_n(v), \tag{2.23}$$

where for $|v| > 1$,

$$G_n(v) = \frac{\Gamma(n+1)\,\Gamma(n+\lambda+a-\alpha_{Q+1})\, v^{-n-\lambda}}{\Gamma(2n+\lambda+1)\,\Gamma(n+\lambda+a+1-\beta_Q)}$$
$$\times {}_{q+2}F_{q+1}\left(\begin{matrix} n+\lambda+a-\alpha_{Q+2} \\ 2n+\lambda+1, n+\lambda+a+1-\beta_Q \end{matrix} \,\middle|\, \frac{-1}{v}\right),$$

$$M_{n,k}(v) = \frac{\Gamma(\beta_k-\alpha_{Q+2})(n+\lambda)_{a-\beta_k}}{\Gamma(\beta_k+1-\beta_Q)(n+1)_{-a+\beta_k}} \tag{2.24}$$
$$\times {}_{q+3}F_{q+2}\left(\begin{matrix} 1, \beta_k-\alpha_{Q+2} \\ \beta_k+n+1-a, \beta_k+1-n-\lambda-a, \beta_k+1-\beta_Q \end{matrix} \,\middle|\, \frac{-1}{v}\right),$$

$$F_k^*(v) = \frac{(-1)^{q+a}(\alpha_{Q+1})_{1-a}\,\Gamma(\beta_k-\beta_Q^{*k})}{\Gamma(2-\beta_k)(\beta_Q)_{-a}\,\Gamma(\beta_k-\alpha_{Q+1})}$$
$$\times {}_{q+2}F_{q+1}\left(\begin{matrix} 1, -\beta_k+1+\alpha_{Q+1} \\ -\beta_k+2, -\beta_k+1+\beta_Q \end{matrix} \,\middle|\, \frac{-1}{v}\right).$$

Proof. If in the definition of $S_n(v)$, one substitutes for $H_n(v)$ and $K_n(v)$ the representations (2.7) and (2.11) developed in Theorems 3 and 4, one obtains an expansion for $S_n(v)$ which agrees with (2.19) in the $L_{n,k}(ve^{i\pi})$ and $G_n(ve^{i2\pi})$ terms, but whose $G_n(v)$ coefficient is $A_2[F(v) - F(ve^{i2\pi})]$. If $\mathrm{Re}\,\sigma > 0$ and $v = e^{-i\pi}$, it was shown in (1.9) that $F(e^{i\pi}) = F(e^{-i\pi})$, so that this $G_n(v)$ coefficient is zero, and the resulting expansion for $S_n(e^{-i\pi})$

reduces to (2.20). More generally, we proceed as follows. From (2.6), it follows that

$$\frac{(-1)^a\,\Gamma(-a+1+\alpha_{Q+1})\,\Gamma(s+\beta_Q)\,\Gamma(1-s-\beta_Q)}{\Gamma(-a+\beta_Q)\,\Gamma(s+1+\alpha_{Q+1})\,\Gamma(-s-\alpha_{Q+1})}$$
$$=(-1)^a\sum_{k=1}^{q} B_k\Gamma(s+\beta_k)\,\Gamma(1-s-\beta_k)+A_1e^{i\pi s}+A_2e^{-i\pi s}. \quad (2.25)$$

Multiplying this identity by

$$\frac{1}{2\pi i}\,\frac{\Gamma(-s)\,\Gamma(s+\alpha_{Q+1})}{\Gamma(s+\beta_Q)}\,(ve^{i\pi})^{-s}$$

and integrating the resulting identity with respect to s over $\mathscr{L}_-$, an s contour of the form L_- which has no poles of $\Gamma(-s)\,\Gamma(1-s-\beta_Q)$ in its interior, we obtain the relationship

$$0=A_1F(v)+A_2F(ve^{i2\pi})+(-1)^a\sum_{k=1}^{q} B_kE_k(v),$$

which can be solved for $A_2F(ve^{i2\pi})$. Substituting this expression for $A_2F(ve^{i2\pi})$ into the above, derived expansion for $S_n(v)$, one obtains (2.19). The asymptotic relations (2.21) and (2.22) follow directly from Theorems 1 and 3.

For $|v|>1$, it follows from (1.3) that $G_n(v)$ has the representation quoted in the theorem, that

$$L_{n,k}(ve^{i\pi})=(-1)^a\,\Gamma(\beta_k)\,\Gamma(1-\beta_k)\,G_n(v)+(ve^{i\pi})^{a-\beta_k}M_{n,k}(v),$$

and that

$$F(v)-\frac{E_k(v)}{\Gamma(\beta_k)\,\Gamma(1-\beta_k)}$$
$$=(-1)^{q+a}\,\frac{\Gamma(-a+\beta_Q)\,\Gamma(\beta_k-\alpha_{Q+1})\,\Gamma(1-\beta_k+\alpha_{Q+1})(ve^{i\pi})^{-1+\beta_k}}{\Gamma(-a+1+\alpha_{Q+1})\,\Gamma(\beta_k-\beta_Q^{*k})\,\Gamma(1-\beta_k+\beta_Q)}\,F_k^*(v).$$

Substituting these expressions into (2.19), we obtain, after some cancellation the result

$$S_n(v)=\sum_{k=1}^{q} v^{a-1}F_k^*(v)\,M_{n,k}(v)$$
$$+\left\{A_1+A_2+(-1)^a\sum_{k=1}^{q} B_k\Gamma(\beta_k)\,\Gamma(1-\beta_k)\right\}F(v)\,G_n(v).$$

The constant coefficient of $F(v)\,G_n(v)$ in this expansion can be deduced in closed form from (2.25) with $s=-a$. This leads to (2.23). ∎

Remark 5. An alternate interpretation of Theorems 3, 4 and 5 is the following. Let $\mathscr{M}_n(\gamma)$ be the linear difference operator of order $q+2$,

$$\mathscr{M}_n(\gamma)=\mathscr{U}_n(\lambda-q-1,0)\prod_{j=1}^{q+1}\mathscr{U}_n(\lambda-q-1+j,\alpha_j)$$
$$-\gamma n(n+\lambda-q-2)\,\mathscr{E}^{-1}\prod_{j=1}^{q}\mathscr{U}_n(\lambda-q+j,\beta_j),$$

where $\prod_{j=1}^{r}P_j=P_1P_2\cdots P_r$, $\mathscr{E}^{-j}$ is the shift operator on n, i.e., $\mathscr{E}^{-j}y_n=y_{n-j}$, and

$$\mathscr{U}_n(\lambda,\mu)=\frac{(n+\lambda-1)(n+\mu)}{2n+\lambda-1}\,\mathscr{E}^0-\frac{n(n+\lambda-1-\mu)}{2n+\lambda-1}\,\mathscr{E}^{-1}.$$

From the simple computation

$$\mathscr{U}_n(\lambda,\mu)\left\{\frac{(n+\lambda)_s}{(n+1)_{-s}}\right\}=\frac{(n+\lambda-1)_s}{(n+1)_{-s}}(s+\mu),$$

it follows that

$$\mathscr{M}_n(\gamma)\left\{\frac{(n+\lambda)_s}{(n+1)_{-s}}\right\}=\frac{(n+\lambda-q-2)_s}{(n+1)_{-s}}(s-a+\alpha_{Q+2})_1$$
$$-\gamma\frac{(n+\lambda-q-2)_{s+1}}{(n+1)_{-s-1}}(s-a+\beta_Q)_1. \quad (2.26)$$

The standard Mellin–Barnes integral representations for the functions $L_{n,k}(ve^{i\pi})$ $(k=1,\dots,q)$, $G_n(v)$ and $G_n(ve^{i2\pi})$ with s as the integration variable, contain the function $(n+\lambda)_s/(n+1)_{-s}$ as a kernel. Using (2.26), it can be shown that $\mathscr{M}_n(v)$ annihilates these integrals.

Thus, the functions

$$\mathscr{B}=\{L_{n,k}(ve^{i\pi})\ (k=1,\dots,q),\ G_n(v),\ G_n(ve^{i2\pi})\},$$

where $G_n(ve^{i2\pi})$ is $G_n(v)$ analytically continued along a contour which encloses $v=0$ but not $v=-1$, satisfy the linear difference equation of order $q+2$,

$$\mathscr{M}_n(v)\{y_n(v)\}=0. \quad (2.27)$$

In fact, Theorem 1 shows that as functions of n, $\mathscr{B}$ is a linearly independent set, and hence that $\mathscr{B}$ is a global basis of (2.27). In particular, Theorems 3 and 4 then imply that the polynomials $H_n(v)$ and $K_n(v)$ also satisfy (2.27).

It can also be shown, that for $|v| < 1$ the functions

$$\mathscr{B}_0 = \{G_{n,r}(v)\ (r = 1,..., q+2) \text{ as defined in } (2.14)\}$$

form a local basis of (2.27), while for $|v| > 1$ the functions

$$\mathscr{B}_\infty = \{G_n(v), G_n(ve^{i2\pi}), M_{n,k}(v)\ (k = 1,..., q) \text{ as defined in } (2.24)\}$$

form a local basis of (2.27). Theorem 5 is then seen to be just representing the solution $S_n(v)$ of (2.27) in terms of the bases $\mathscr{B}$ and $\mathscr{B}_\infty$.

Remark 6. Some insight into the best choice of λ can be deduced from (2.22), which can be rewritten in the form

$$\frac{S_n(-1)}{H_n(-1)} = \mathscr{O}(n^{-\lambda-\omega}), \qquad n \to \infty,$$

where the constant ω is independent of λ and the parameter a. Hence, the bound for $S_n(-1)/H_n(-1)$ is smallest when λ is largest. In view of the conditions

$$a + \lambda \leqslant q + 1, \qquad 0 \leqslant a,$$

this means that one should take $a = 0$, $\lambda = q + 1$—at least at $v = -1$, the point of "worst" convergence.

3. General Case, $p \leqslant q + 1$

Our main result is the following.

THEOREM 6. *Let* $p+1, q+1, n+1, a+1, \lambda+1$ *be positive integers,* α_r $(r = 1,..., p+1;\ \alpha_{p+1} = a)$, β_k $(k = 1,..., q)$ *be complex numbers such that*

$$a + \lambda = \textit{an integer} \leqslant q + 1;$$

$$-1 + \beta_k,\ a - \beta_k,\ -1 - a + \beta_k \neq \textit{a negative integer},\ k = 1,..., q;$$

$$\beta_k - \beta_j \neq \textit{an integer},\ k \neq j, \quad k, j = 1,..., q;$$

$$-1 + \alpha_k,\ -a + \alpha_k \neq \textit{a negative integer},\ k = 1,..., p,$$

and set

$$H_n^{p,q}(v) = {}_{q+2}F_p\left(\begin{matrix}-n, n+\lambda, -a+\beta_Q\\ -a+1+\alpha_P\end{matrix}\,\middle|\, -v\right),$$

$$K_n^{p,q}(v) = \sum_{k=a}^{n} (-v)^k \sum_{j=0}^{n-k} \frac{(-n)_{k+j}(n+\lambda)_{k+j}(-a+\beta_Q)_{k+j}(\alpha_P)_j}{(-a+1+\alpha_P)_{k+j}(k+j)!(\beta_Q)_j j!}.$$

Then for $p \leqslant q+1$ ($|v|>1$ *if* $p=q+1$),

$$\begin{aligned}
S_n^{p,q}(v) &= {}_pF_q\left(\begin{matrix}\alpha_P\\ \beta_Q\end{matrix}\,\middle|\,\frac{-1}{v}\right) H_n^{p,q}(v) - K_n^{p,q}(v),\\
&= \frac{\Gamma(a+1-\beta_Q)}{\Gamma(a-\alpha_P)}\, {}_pF_q\left(\begin{matrix}\alpha_P\\ \beta_Q\end{matrix}\,\middle|\,\frac{-1}{v}\right)\\
&\quad\times \frac{\Gamma(n+1)\,\Gamma(n+\lambda+a-\alpha_P)[(-1)^{q-p+1}v]^{-(n+\lambda)}}{\Gamma(n+\lambda+a+1-\beta_Q)\,\Gamma(2n+\lambda+1)}\\
&\quad\times {}_{p+1}F_{q+1}\left(\begin{matrix}n+\lambda, n+\lambda+a-\alpha_P\\ 2n+\lambda+1, n+\lambda+a+1-\beta_Q\end{matrix}\,\middle|\,\frac{(-1)^{q-p}}{v}\right)\\
&\quad+ \sum_{k=1}^{q} \frac{(\alpha_P)_{1-a}}{(\beta_Q)_{-a}}\, \frac{\Gamma(-a+\beta_k)}{\Gamma(2-\beta_k)}\, \frac{(-v)^{a-1}}{(\beta_Q^{*k}-\beta_k)}\\
&\quad\times {}_{p+1}F_{q+1}\left(\begin{matrix}1, -\beta_k+1+\alpha_P\\ -\beta_k+2, -\beta_k+1+\beta_Q\end{matrix}\,\middle|\,\frac{-1}{v}\right) \frac{(n+\lambda)_{a-\beta_k}}{(n+1)_{-a+\beta_k}}\\
&\quad\times {}_{p+2}F_{q+2}\left(\begin{matrix}1, \beta_k-a, \beta_k-\alpha_P\\ \beta_k+n+1-a, \beta_k+1-n-\lambda-a, \beta_k+1-\beta_Q\end{matrix}\,\middle|\,\frac{(-1)^{q-p}}{v}\right).
\end{aligned} \tag{3.1}$$

Proof. If $p=q+1$, (3.1) reduces to (2.23) in Theorem 5. The other cases are confluent limiting forms of this $p=q+1$ case. To see this, let $\sigma=\alpha_{q+1}$. Then

$$\lim_{\sigma\to\infty} H_n^{q+1,q}(\sigma v) = H_n^{q,q}(v), \qquad \lim_{\sigma\to\infty} K_n^{q+1,q}(\sigma v) = K_n^{q,q}(v),$$

$$\lim_{\sigma\to\infty} S_n^{q+1,q}(\sigma v) = S_n^{q,q}(v),$$

which is the $p=q$ case. After $q-p+1$ such limits, one arrives at the general statement of the theorem. ∎

Remark 7. Preliminary computations indicate that for v ($\neq 0$) fixed,

$$H_n^{p,q}(v) \sim \frac{(n+\lambda)_n(-a+\beta_Q)_n}{(-a+1+\alpha_P)_n} v^n e^{b/2v}, \quad n \to \infty; \tag{3.2}$$

$${}_{p+1}F_{q+1}\left(\begin{matrix} n+\lambda, n+\lambda+a-\alpha_P \\ 2n+\lambda+1, n+\lambda+a+1-\beta_Q \end{matrix} \,\middle|\, \frac{(-1)^{q-p}}{v}\right) \sim e^{b/2v}, \quad n \to \infty; \tag{3.3}$$

$$\frac{(n+\lambda)_{a-\beta_k}}{(n+1)_{-a+\beta_k}} {}_{p+2}F_{q+2}\left(\begin{matrix} 1, \beta_k - a, \beta_k - \alpha_P \\ \beta_k+n+1-a, \beta_k+1-n-\lambda-a, \beta_k+1-\beta_Q \end{matrix} \,\middle|\, \frac{(-1)^{q-p}}{v}\right)$$
$$\sim n^{2a-2\beta_k}, \qquad n \to \infty; \tag{3.4}$$

$$p \leqslant q, \qquad b = 1 \text{ if } p = q, \qquad b = 0 \text{ if } p < q,$$

which would imply

$$\lim_{n\to\infty} \frac{K_n^{p,q}(v)}{H_n^{p,q}(v)} = {}_pF_q\left(\begin{matrix}\alpha_P \\ \beta_Q\end{matrix} \,\middle|\, \frac{-1}{v}\right), \qquad p \leqslant q+1. \tag{3.5}$$

Rigorous derivation of (3.2)–(3.4) would then establish (3.5).

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Higher Transcendental Functions," Vol. 1, McGraw–Hill, New York, 1953.
2. Y. L. Luke, "The Special Functions and their Approximations," Vols. 1 and 2, Academic Press, New York, 1969.
3. Y. L. Luke, "Algorithms for the Computation of Mathematical Functions," Academic Press, New York, 1977.
4. J. L. Fields, Rational approximations .to generalized hypergeometric functions, *Math. Comput.* **19** (1965), 606–624.
5. J. L. Fields, A linear scheme for rational approximations, *J. Approx. Theory* **6** (1972), 161–175.
6. J. L. Fields, Uniform asymptotic expansions of a class of Meijer *G*-functions for a large parameter, *SIAM J. Math. Anal.*, in press.
7. N. E. Nørlund, Hypergeometric functions, *Acta Math.* **94** (1955), 289–349.